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The Fermat–Weber Problem and Inner Product Spaces

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In the setting of real normed spaces, we study the Fermat–Weber problem which deals with the minimization of the sum of weighted distances from a variable point to the points of a given finite set A. With techniques of best approximation we obtain a description of the set of solutions to this problem. Then we characterize inner product spaces as spaces in which the set of solutions to such problems meets the affine hull of A. The major tool is a characterization of inner product spaces, with finite dimension at least three, lying on some property of the exposed points of the unit ball.

1. INTRODUCTION

The Fermat–Weber problem is an optimization problem associated with a real normed space X, a finite subset A of X having at least two points, and a family $w=(w_a)_{a\in A}$ of positive weights. The function F to minimize is defined on X by $F(x)=\sum_{a\in A} w_a \|x-a\|$. The set of minimizers of F on X is a (possible empty) bounded closed convex subset of X, denoted by $M_w(A)$. Obviously if $w'_a=kw_a$ with $k>0$, then $M_{w'}(A)=M_w(A)$. When $w_a=1$ for each a, the problem is called the Fermat (location) problem and the set $M_w(A)$ is denoted by $M_1(A)$. The names of Steiner and Lamé are also associated with this problem (e.g., [7]).

If X is an inner product space, or if X is two dimensional and whatever the norm is, then $M_w(A)$ intersects conv(A), *the convex hull of* A, for every subset A and every family w [11, 14]. In the paper we study some reciprocal properties with X *of dimension at least three.* We state for instance that if, for every A and every w, $M_w(A)\cap \text{conv}(A)\neq\varnothing$, then X is an inner product space. The same conclusion holds if, for every A, $M_1(A)\cap \text{aff}(A)\neq\varnothing$, where aff$(A)$ denotes *the affine hull* (or span) of A. This last result shows that an assumption made in [4], according to which $M_1(A)$ intersects always aff(A), cannot be valid.

We call *hull property* for the Fermat (or the Fermat–Weber) problem the fact that $M_1(A)$ (or $M_w(A)$) intersects the convex or the affine hull of A,

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for every A (and every w). In Section 2 we show that these four hull properties are equivalent. Then, in order to obtain the main results, we give in Section 3 a geometrical description of the set $M_w(A)$ and in Section 4, a characterization of inner product spaces. This characterization is valid for spaces of finite dimension, at least three. It lies on a property of the set of *exposed points* of the unit ball. Section 5 presents the main results. The idea is roughly to prove that if X, of dimension at least three, is not an inner product space, then a finite subset $\tilde{A}$ and a family of weights w exist such that $\tilde{A}$ is included in an open halfspace and the intersection of $M_w(\tilde{A})$ with the affine hull of $\tilde{A}$ is either empty or reduced to $\{0\}$. This contradicts some hull property.

Theorem 1 of this paper is more or less known. A partial proof is presented in [4]. Theorem 2 is given in [5], but a proof using best approximation theory seems new. To the best of our knowledge a characterization of finite dimensional inner products spaces, as stated in Theorem 3, lying on properties of exposed points of the unit ball and normal cones at these points, cannot be found in the literature. Nothing similar is in [1]. Besides, the so-called Geometric Lemma in Section 4.3 is a refinement of Straszewicz's Theorem. Finally, Theorem 4 and its Corollaries are answers to natural questions. They are related to known properties of Chebyshev centers [6, 9], but cannot apparently be deduced from them. We tackle our problem by a completely different method.

2. Interrelations between Hull Properties

THEOREM 1. *Let A denote a finite subset of X with at least two points and $w = (w_a)_{a \in A}$ a family of positive numbers. The following are equivalent*:

(i) *for every A and every w,* $M_w(A) \cap \operatorname{conv}(A) \neq \varnothing$;

(ii) *for every A and every w,* $M_w(A) \cap \operatorname{aff}(A) \neq \varnothing$;

(iii) *for every A,* $M_1(A) \cap \operatorname{conv}(A) \neq \varnothing$;

(iv) *for every A,* $M_1(A) \cap \operatorname{aff}(A) \neq \varnothing$.

Proof. It is sufficient to prove (ii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i). Let A and w be such that $K = M_w(A) \cap \operatorname{aff}(A)$ is nonempty. Suppose $K \cap \operatorname{conv}(A) = \varnothing$. These two disjoint nonempty compact convex sets can be strictly separated by a closed affine hyperplane H. Let $b \in K$. For each $a \in A$, we call $g(a)$ the intersection of the line joining a and b with H and we let $A' = g(A)$. For each $a' \in A'$, we define $\lambda_{a'}$ by

$$\lambda_{a'} = \sum_{\{a;\, g(a) = a'\}} w_a.$$

We call $M_\lambda(A')$ the set of solutions to the problem $\min_{x\in X}\sum_{a'\in A'}\lambda_{a'}\|x-a'\|$. By hypothesis, $M_\lambda(A')\cap\operatorname{aff}(A')$ is nonempty.

By summing, for all $a'\in A'$, the following,

$$\sum_{\{a;\, g(a)=a'\}} w_a\|b-a\| = \lambda_{a'}\|b-a'\| + \sum_{\{a;\, g(a)=a'\}} w_a\|a'-a\|,$$

we get

$$\sum_{a\in A} w_a\|b-a\| = \sum_{a'\in A'} \lambda_{a'}\|b-a'\| + \sum_{a\in A} w_a\|g(a)-a\|.$$

We now choose $b'\in M_\lambda(A')\cap\operatorname{aff}(A')$. Then $b'\in H\cap\operatorname{aff}(A)$ and we have, by using $\|g(a)-a\|\geqslant\|b'-a\|-\|g(a)-b'\|$,

$$\sum_{a\in A} w_a\|b-a\| \geqslant \sum_{a'\in A'} \lambda_{a'}\|b-a'\| - \sum_{a\in A} w_a\|g(a)-b'\| + \sum_{a\in A} w_a\|b'-a\|.$$

Since

$$\sum_{a'\in A'} \lambda_{a'}\|b-a'\| \geqslant \sum_{a'\in A'} \lambda_{a'}\|b'-a'\|$$

and

$$\sum_{a\in A} w_a\|g(a)-b'\| = \sum_{a'\in A'} \lambda_{a'}\|b'-a'\|,$$

we get

$$\sum_{a\in A} w_a\|b-a\| \geqslant \sum_{a\in A} w_a\|b'-a\|,$$

which entails $b'\in M_w(A)$. Thus $b'\in K\cap H$, which is impossible. This contradiction means $K\cap\operatorname{conv}(A)\neq\varnothing$, i.e., $M_w(A)\cap\operatorname{conv}(A)\neq\varnothing$.

The idea of this proof is used in [4] to establish (iv) $\Rightarrow$ (iii).

(iv) $\Rightarrow$ (ii) Suppose (iv) holds true and let A be given.

Note initially that A can be assumed included in a ball $B(0,r)=\{x;\|x\|\leqslant r\}$. Then $M_w(A)$ is included in $B(0,2r)$ for each w. Indeed let $x\notin B(0,2r)$: for every $a\in A$, we have $\|x-a\|\geqslant|\|x\|-\|a\||>r\geqslant\|a\|$ and then

$$\sum_{a\in A} w_a\|x-a\| > \sum_{a\in A} w_a\|a\|.$$

This implies clearly $M_w(A)\subseteq B(0,2r)$.

Now as a first step we prove that $M_w(A) \cap \text{aff}(A) \neq \varnothing$ for integer weights. We first consider the case where, for one $\bar{a} \in A$, $w_{\bar{a}} = k$, k integer, $k \geqslant 2$ and $w_a = 1$ for each $a \neq \bar{a}$. Let $(a_1^n), ..., (a_k^n)$ be k sequences of points in $\text{aff}(A) \cap B(0, r)$ such that (1) $\lim_n a_1^n = \cdots = \lim_n a_k^n = \bar{a}$, (2) for each n, the k points $a_1^n, ..., a_k^n$ are distinct and do not belong to $A \setminus \{\bar{a}\}$. To the set $A_n = (A \setminus \{\bar{a}\}) \cup \{a_1^n, ..., a_k^n\}$, we associate the function F_n,

$$F_n(x) = \sum_{b \in A_n} \|x - b\| = \sum_{a \in A,\, a \neq \bar{a}} \|x - a\| + \sum_{1 \leqslant j \leqslant k} \|x - a_j^n\|,$$

and we let

$$F(x) = \sum_{a \in A,\, a \neq \bar{a}} \|x - a\| + k\|x - \bar{a}\|.$$

Since $A_n \subset B(0, r)$ and $\text{aff}(A_n) \subset \text{aff}(A)$, the optimization problem $\min_{x \in X} F_n(x)$ has a solution x_n in the compact set $\text{aff}(A) \cap B(0, 2r)$. The sequence (x_n) has a subsequence, yet denoted by (x_n), which converges to $x \in \text{aff}(A) \cap B(0, 2r)$. Since F_n converges to F uniformly on $B(0, 2r)$, $F_n(x_n)$ converges to $F(x)$ and $x \in \text{aff}(A)$ is a solution to the problem $\min_{x \in X} F(x)$. The same reasoning works to obtain $M_w(A) \cap \text{aff}(A) \neq \varnothing$ for integer weights w_a, and then for rational weights.

As a second step we prove that if $M_w(A) \cap \text{aff}(A) \neq \varnothing$ for rational weights, then the same is true for real weights. Indeed if $(w_a)_{a \in A}$ is a family of real positive numbers, we choose, for each a, a sequence $(w_a^{(n)})$ of rational positive number which converges to w_a. With $F_n(x) = \sum_{a \in A} w_a^{(n)} \|x - a\|$ and $F(x) = \sum_{a \in A} w_a \|x - a\|$, the reasoning is the same as in the first step. The result follows. ∎

3. A Geometrical Description of the Set of Solutions to a Fermat–Weber Problem

A geometrical description of the set of solutions to a Fermat–Weber problem may be obtained with techniques from convex analysis (see[5]). We present here a procedure based on the theory of *best approximation*.

In order to express more easily the problem in the context of best approximation, we give the finite set A by $A = \{a_1, ..., a_m\} (m \geqslant 2)$ and we let $w_i = w_{a_i} (1 \leqslant i \leqslant m)$. We denote by Ξ the space X^m equipped with the norm.

$$|||\xi||| = |||(x_1, ..., x_m)||| = \sum_{i=1}^{m} \|x_i\|.$$

The subspace Λ of Ξ is defined by

$$\Lambda = \{\eta = (w_1 x, ..., w_m x);\ x \in X\}.$$

According to the initial definition, $x \in X$ belongs to $M_w(A)$ if and only if $\eta = \{w_1 x, ..., w_m x)$ is a *best approximationt* to $\beta = (w_1 a_1, ..., w_m a_m)$ from Λ in Ξ. Clearly β does not belong to Λ. We can use a classical characterization of the set $P_\Lambda(\beta)$ of best approximants to β from Λ [12].

To that end we mention that the dual space Ξ^* of Ξ is $(X^*)^m$ endowed with the norm

$$|||\varphi||| = |||(f_1, ..., f_m)||| = \max_{1 \leqslant i \leqslant m} \|f_i\|,$$

where the norm on the dual X^* of X is denoted by $\|\cdot\|$. The pairing between Ξ and Ξ^* is defined for $\varphi = (f_1, ..., f_m) \in \Xi^*$ and $\xi = (x_1, ..., x_m) \in \Xi$ by

$$\langle\langle \varphi, \xi \rangle\rangle = \sum_{i=1}^{m} \langle f_i, x_i \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between X and X^*. The condition $\varphi \in \Lambda^\perp$ means $\langle\langle \varphi, \eta \rangle\rangle = 0$ for each $\eta \in \Lambda$, what is equivalent, for $\varphi = (f_1, ..., f_m)$, to $\sum_{i=1}^m w_i f_i = 0$. Let us introduce again a notation. To each $\varphi \in \Xi^*$, we associate the subset Γ_φ of Λ, which depends on β,

$$\Gamma_\varphi = \{\eta \in \Lambda;\ \langle\langle \varphi, \beta - \eta \rangle\rangle = |||\beta - \eta|||\}.$$

Then we have the results:

1. If $\varphi \in \Xi^*$ satisfies $|||\varphi||| = 1$, $\varphi \in \Lambda^\perp$ and $\Gamma_\varphi \neq \emptyset$, then $\Gamma_\varphi = P_\Lambda(\beta)$.

2. If $P_\Lambda(\beta)$ is nonempty, then there exists $\varphi \in \Xi^*$ satisfying $|||\varphi||| = 1$, $\varphi \in \Lambda^\perp$ such that $P_\Lambda(\beta) = \Gamma_\varphi$.

Our task is now to transpose these results in the space X. Let $\varphi = (f_1, ..., f_m) \in \Xi^*$ be such that $|||\varphi||| = \max_{1 \leqslant i \leqslant m} \|f_i\| = 1$. Then $\eta = (w_1 x, ..., w_m x)$ is a member of Γ_φ if and only if

$$\sum_{i=1}^{m} w_i \langle f_i, a_i - x \rangle = \sum_{i=1}^{m} w_i \|a_i - x\|.$$

Since $\|f_i\| \leqslant 1$ for each i, that is equivalent to

$$\forall i = 1, ..., m \qquad \langle f_i, a_i - x \rangle = \|a_i - x\|.$$

These conditions are profitably expressed with some cones. For $f \in X^*$, $\|f\| \leqslant 1$, we let

$$N(f) = \{z \in X;\ \langle f, z \rangle = \|z\|\}.$$

If $\|f\| < 1$, then $N(f) = \{0\}$. If $\|f\| = 1$, $N(f)$ is the (possible empty) convex cone generated by the face of the unit ball of X given by $\{x; \|x\| = 1, \langle f, x \rangle = 1\}$. We can also define $N(f)$ as the normal cone to the unit ball of X^* at f. Hence, for $|||\varphi||| = 1$, $(w_1 x, ..., w_m x)$ is a member of Γ_φ if and only if

$$x \in \bigcap_{i=1}^{m} (a_i - N(f_i)).$$

The theorem sums up the preceding discussion.

THEOREM 2. *Let $A = \{a_1, ..., a_m\} \subset X$ and let $w_i > 0$ $(1 \leqslant i \leqslant m)$.*

1. *If $(f_1, ..., f_m) \in (X^*)^m$ satisfies $\max_{1 \leqslant i \leqslant m} \|f_i\| = 1$, $\sum_{i=1}^m w_i f_i = 0$, and $\bigcap_{i=1}^m (a_i - N(f_i)) \neq \varnothing$, then*

$$\bigcap_{i=1}^{m} (a_i - N(f_i)) = M_w(A).$$

2. *If $M_w(A)$ is nonempty, then there exists $(f_1, ..., f_m) \in (X^*)^m$ satisfying $\max_{1 \leqslant i \leqslant m} \|f_i\| = 1$ and $\sum_{i=1}^m w_i f_i = 0$ such that*

$$M_w(A) = \bigcap_{i=1}^{m} (a_i - N(f_i)).$$

Paper [3] gives sufficient and necessary conditions such that z is a point of $M_1(A)$ with restrictive assumptions: 1. z does not belong to A nor is on any line determined by points of A, 2. X is finite dimensional and its unit ball is smooth and rotund. Actually Theorem 2 implies these results of [3], as a particular case.

Remark 1. As an immediate consequence of Theorem 2, we can give some information about unicity of the solution to a Fermat–Weber problem. An obvious sufficient condition such that $M_w(A)$ contains no more than one point, is that it is defined as $\bigcap_{i=1}^m (a_i - N(f_i))$, with at least two cones $N(f_i)$ being non-colinear halflines. In fact $N(f)$ is a halfline if $\|f\| = 1$ and if the hyperplane $\langle f, z \rangle = 1$ meets the unit ball of X at one point.

Remark 2. Let $\rho_i (1 \leqslant i \leqslant m)$ be positive numbers. If we have $a_i \in N(f_i)$ for $f_i \in X^*$, $\|f_i\| \leqslant 1$, then $\rho_i a_i \in N(f_i)$. It follows from Theorem 2 that, if $\tilde{A}$ is the set $\{\rho_1 a_1, ..., \rho_m a_m\}$, then $0 \in M_w(A)$ if and only if $0 \in M_w(\tilde{A})$.

4. Characterization of Inner Product Spaces with Finite Dimension at Least Three

The results of this section seem to have their own interest.

4.1. *Notations and First Results*

In the whole section, X is a real *finite dimensional* space, of dimension l, in which B and S are the unit ball and the unit sphere. In the dual space X^*, B^* and S^* are the unit ball and the unit sphere. The pairing between X and X^* is denoted by $\langle \cdot, \cdot \rangle$. Let J denote the *duality mapping* for X:

$$J(x) = \{p \in X^*;\ \langle p, x \rangle = \|p\|\ \|x\| \quad \text{and} \quad \|p\| = \|x\|\}.$$

For $x \in S$, $J(x)$ is nothing else than the subdifferential of the norm $\|\ \|$ at x.

If H is a hyperplane of X, then H^+ denotes any one of the two open halfspaces defined by H. If D is a convex subset of X, then $\text{ri}(D)$ denotes the relative interior of D, i.e., the interior of D in $\text{aff}(D)$.

Let C be a bounded closed convex subset of X and suppose $x \in C$. Then x is called an *extreme point* of C if $x = (1-t)\,y + tz$ with $0 < t < 1, y \in C$ and $z \in C$, entails $y = z$. The point x is called an *exposed point* of C if there is $p \in E^*$ such that $\langle p, x \rangle > \langle p, y \rangle$ whenever $x \neq y$ and $y \in C$. The linear functional p is said to *expose* x in C. The normal cone to C at x, $N_C^*(x)$, is defined by

$$N_C^*(x) = \{p \in X^*; \forall y \in C,\ \langle p, x - y \rangle \leqslant 0\}.$$

If x belongs to the boundary of C, then $N_C^*(x)$ generates a vector space of dimension at least 1. If this dimension is l, then x called a *vertex* of C. We denote the set of extreme points of C by $\text{ext}(C)$, the set of exposed points of C by $\exp(C)$, and the set of vertices of C by $\text{ver}(C)$. We obviously have $\text{ver}(C) \subset \exp(C) \subset \text{ext}(C)$.

Remark 3. If x is an exposed point of B and if $p \in S^*$ belongs to $\text{ri}(N_B^*(x))$, then the cone $N(p) = \{y \in X;\ \langle p, y \rangle = \|y\|\}$ is reduced to the halfline with origin at 0 and passing through x.

4.2. *Theorem*

The following theorem gives a characterization of inner product spaces with finite dimension at least three, depending on some property of the exposed points of the unit ball.

Theorem 3. *Let X be as in Section* 4.1. *with $l \geqslant 3$. The following are equivalent*:

(i) *X is an inner product space, i.e., its norm is deduced from an inner product on X*;

(ii) *for every hyperplane H and for every finite subset A included in $H^+ \cap \exp(B)$, we have*

$$0 \notin \operatorname{conv}\Big(\bigcup_{a \in A} \operatorname{ri}(J(a))\Big);$$

(iii) *for every hyperplane H and for every finite subset A included in $H^+ \cap \exp(B)$, we have*

$$0 \notin \operatorname{conv}\Big(\bigcup_{a \in A} J(a)\backslash\{0\}\Big).$$

Remark 4. For $a \in S$, $N_B^*(a)$ is the cone generated by the subset $J(a)$. Thus condition in (ii) (resp. in (iii)) can be equivalently written $0 \notin \operatorname{conv}(\bigcup_{a \in A} \operatorname{ri} N_B^*(a))$ (resp. $0 \notin \operatorname{conv}(\bigcup_{a \in A} N_B^*(a)\backslash\{0\})$.

Remark 5. Such a characterization of inner product spaces is not valid in an infinite dimensional space. Indeed, in $c_0(\mathbb{N})$, $\exp(B)$ is empty and therefore statements (ii) and (iii) of Theorem 3 are true. If X is two-dimensional, then statements (ii) and (iii) of Theorem 3 are always true.

In Theorem 3, the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (ii) are obvious. In order to prove the implications (ii) $\Rightarrow$ (i), we need some preliminary results. We give now a Geometrical Lemma and a Corollary, and we will continue the proof of Theorem 3 in Section 4.4.

4.3. *A Geometrical Lemma*

It is well known that the set $\exp(C)$ is dense in $\operatorname{ext}(C)$ (see [13, Thm. 11.6]). A stronger approximation result can be obtained with $\exp(C)$ in $\operatorname{ext}(C)\backslash\operatorname{ver}(C)$, which concerns also the normal cones at these points. After submitting this paper, I learned that a result of the same nature had been obtained independently by Klee [10].

GEOMETRICAL LEMMA. *Let X be finite-dimensional. Let C be a bounded closed convex subset of X with a nonempty interior. Let $x_0 \in \operatorname{ext}(C)\backslash\operatorname{ver}(C)$, $p_0 \in N_C^*(x_0) \cap S^*$, and $\varepsilon > 0$. Then there exist $y \in \exp(C)$, $y \neq x$, and $q \in N_C^*(y) \cap S^*$ such that*

$$\|x_0 - y\| < \varepsilon \qquad \textit{and} \qquad \|p_0 - q\| < \varepsilon.$$

Obviously the above property is not satisfied if x_0 is a vertex of C.

Proof. We suppose that X is identified with $\mathbb{R}^l$, endowed with its canonical scalar product. The words "projection" and "orthogonal" must be understood in reference to this scalar product.

We assume $x_0 = 0$ and we denote by k the dimension of the vector space generated by $N_C^*(0)$. Since $0 \in \mathrm{ext}(C) \backslash \mathrm{vert}(C)$, we have $1 \leqslant k \leqslant l-1$. It is sufficient to consider $p_0 \in \mathrm{ri}(N_C^*(0)) \cap S^*$.

Let G be the hyperplane $\langle p_0, z \rangle = 0$ and let L be the projection of $N_C^*(0)$ on G. Since p is in $\mathrm{ri}(N_C^*(0))$, L is a subspace of G of dimension $k-1$.

The point 0 is an extreme point of the convex set $G \cap C$. Note that if 0 is an exposed point of $G \cap C$, then $G \cap C = \{0\}$. In any case we have $\langle q, x \rangle \leqslant 0$ for each $x \in G \cap C$ and each $q \in L$. Let $q_0 \in G \cap S^*$, orthogonal to L and belonging to $N_{G \cap C}^*(0)$. For $\lambda > 0$, let G_λ be the hyperplane in X orthogonal to the vector $p_0 + \lambda q_0$. Then the closed halfspace G_λ^+ with boundary G_λ, which contains p_0, meets C along a convex set D_λ (a slice of C). The set D_λ has a nonempty interior, otherwise the cone $N_C^*(0)$ is of dimension $k+1$. According to [2], the set of p which expose a point of D_λ is dense in X^*. Thus for $\varepsilon > 0$ there are $y \in D_\lambda \cap G_\lambda^+$ and q, which exposes y in D_λ, such that $\|q - p_0 - \lambda q_0\| < \varepsilon/2$. Clearly q exposes y in C. The family $(D_\lambda)_{\lambda > 0}$ is a nonincreasing family of compact sets such that $\bigcap_{\lambda > 0} D_\lambda = \{0\}$. Hence the diameter of D_λ tends to 0 as λ tends to 0. It is possible to choose $\lambda < \varepsilon/2$ such that the diameter of D_λ is less than ε. Then we have

$$y \neq 0, \qquad \|y\| < \varepsilon, \qquad \|q - p_0\| < \varepsilon,$$

and q exposes y in C. ∎

COROLLARY. *Let X be finite-dimensional. Let C be a bounded closed convex subset of X with a nonempty interior. Then there is a countable subset Δ of* $\mathrm{ext}(C)$ *such that, for each $x \in \mathrm{ext}(C)$, each $p \in N_C^*(x) \cap S^*$, and each ε, there exist $y \in \Delta$ and $q \in N_C^*(y) \cap S^*$ such that*

$$\|x - y\| < \varepsilon \qquad \textit{and} \qquad \|p - q\| < \varepsilon.$$

Proof. We note first that $\mathrm{ver}(C)$ is countable [13, Thm. 11.2]. Then we let

$$R_1 = \{(x, p) \in C \times S^*; x \in \mathrm{ext}(C) \backslash \mathrm{ver}(C), p \in N_C^*(x)\}$$

and

$$R_2 = \{(x, p) \in C \times S^*; x \in \exp(C), p \in N_C^*(x)\}.$$

From the geometric lemma we deduce that, in vicinity of each point of R_1, there is a point of R_2. Since C and S^* are metric compact, for each n we can cover R_1 with a finite numbers of subsets

$$\{(x, p) \in C \times S^*; \|x - x_2\| < \frac{1}{2n}, \|p - p_2\| < \frac{1}{2n}, (x_2, p_2) \in R_2\}.$$

Let Σ_n be the finite set of points $(x_2, p_2) \in R_2$ obtained in this manner. The set

$$\Delta = \operatorname{ver}(C) \cup \left(\bigcup_n \Sigma_n \right)$$

has the desired property. ∎

4.4. *Proof of Theorem* 3

We prove the implication (ii) → (i) in three steps. We use, at the end, a characterization of inner product spaces due to James [8] (see [1, (12.9)]), which is as follows. If, in real normed space X of dimension at least three, for every hyperplane H, there exists $u \neq 0$ such that $\|x + tu\| \geqslant \|x\|$ for each $x \in H$ and $t \in \mathbb{R}$, then X is an inner product space.

Let H be a hyperplane in X. Suppose statement (ii) of Theorem 3 holds true.

First Step. For a start we establish the following result. If (A_n) is an increasing sequence of finite subsets of $\exp(B) \cap H^+$, then there exists $u \in S$ such that, for each $a \in \bigcup_n A_n$ and each $p \in J(a)$, $\langle p, u \rangle \geqslant 0$.

Indeed, from (ii), we deduce, by using a separation theorem, that, for each n there exists $u_n \in S$ such that $\langle p, u_n \rangle \geqslant 0$ for each $a \in A_n$ and each $p \in J(a)$. By compactness of S, there is a subsequence of u_n which converges to $u \in S$. We consider now the subsets (A_n) and the vectors (u_n) associated to this subsequence. Let $a \in \bigcup_n A_n$; then, for n great enough, a belongs to A_n and, for $p \in J(a)$, $\langle p, u_n \rangle \geqslant 0$, hence $\langle p, u \rangle \geqslant 0$.

Second step. We can apply the result of the first step to the countable set introduced in the preceding Corollary. Let us say that $a \in S$ satisfies the *property* $\Pi(u)$ if, *for each* $p \in J(a)$, *we have* $\langle p, u \rangle \geqslant 0$. The first step gives a vector $u \in S$ such that every member of $\Delta \cap H^+$ satisfies $\Pi(u)$.

First every member of $\operatorname{ext}(B) \cap H^+$ satisfies $\Pi(u)$. This is a consequence of the property of Δ given in the Corollary.

Then every member of $S \cap H^+$ satisfies $\Pi(u)$. Indeed, let $a \in S \cap H^+$. According to the Krein–Milman theorem, there exist a finite subset $\{a_1, ..., a_h\}$ of $\operatorname{ext}(B)$ and positive numbers $\alpha_1, ..., \alpha_h$ ($\sum_{i=1}^h \alpha_i = 1$) such that $a = \sum_{i=1}^h \alpha_i a_i$. There is at least one $j (1 \leqslant j \leqslant h)$ with $a_j \in H^+$, otherwise $a \notin H^+$. Let $p \in J(a)$. Then $1 = \langle p, a \rangle = \sum_{i=1}^h \alpha_i \langle p, a_i \rangle$. This entails $\langle p, a_i \rangle = 1$ for each i, i.e., $p \in J(a_i)$. Particularly, $p \in J(a_j)$ where $a_j \in \operatorname{ext}(B) \cap H^+$. Hence, we have $\langle p, u \rangle \geqslant 0$. This means that a satisfies $\Pi(u)$.

Third Step. We know that each member of $S \cap H^+$ satisfies $\Pi(u)$. Let $a \in S \cap H^+$ and $p \in J(a)$. For $\lambda > 0$, we have

$$\|a + \lambda u\| \geqslant \langle p, a + \lambda u \rangle = \langle p, a \rangle + \lambda \langle p, u \rangle = 1 + \lambda \langle p, u \rangle.$$

Thus we have, for $a \in S \cap H^+$ and $\lambda \geqslant 0$, $\|a + \lambda u\| \geqslant 1$. The same is true if a belongs to $S \cap H$. Let $-b \in S \cap H^+$. Then we have, for $\lambda \geqslant 0$, $\|-b + \lambda u\| \geqslant 1$. Hence $\|b - \lambda u\| \geqslant 1$ and the same is true if b belongs to $S \cap H$. Then we have $\|x + tu\| \geqslant 1$, for each $x \in S \cap H$ and $t \in \mathbb{R}$.

This is James' condition, which entails that X is an inner product space.

5. The Main Results

In the whole section X is assumed to be of *dimension at least three.*

Theorem 4. *If, for every subset A with three or four elements and every family $w = (w_a)_{a \in A}$ of positive weights, we have $M_w(A) \cap \operatorname{aff}(A) \neq \emptyset$, then X is an inner product space.*

Proof. Suppose X is not an inner product space. Then X has a three-dimensional subspace Y which is not an inner product space.

First Step. In this first step everything will take place in Y. Let $B(Y)$ be the unit ball of Y. According to Theorem 3 and Remark 4, there exist a hyperplane H in Y and a finite subset A_0 in $\exp(B(Y)) \cap H^+$ such that $0 \in \operatorname{conv}\left(\bigcup_{a \in A_0} \operatorname{ri} N^*_{B(Y)}(a)\right)$. Due to the Caratheodory Theorem, there are a subset A of A_0 with four or fewer points, positive weights w_a, and vectors $p_a \in \operatorname{ri}(N^*_{B(Y)}(a))$, $(a \in A)$ such that $0 = \sum_{a \in A} w_a p_a$. From Remark 3 we deduce that $\bigcap_{a \in A} (a - N(p_a)) = \{0\}$. It follows from Theorem 2 that the problem $\min_{y \in Y} \sum_{a \in A} w_a \|y - a\|$ has a unique solution: $\{0\}$. Choose then $\rho_a > 0$ such that all points $\tilde{a} = \rho_a a$ are in the same affine hyperplane parallel to H and let $\tilde{A}$ be the set for which points are the $\tilde{a}$. Due to Remark 2, the problem $\min_{y \in Y} \sum_{\tilde{a} \in \tilde{A}} w_a \|y - \tilde{a}\|$ has the unique solution: $\{0\}$. Note finally that $\tilde{A}$ cannot be reduced to a singleton or to a pair of points.

Second Step. We are now in the whole space X. Then, for the subset $\tilde{A}$ and the associated weights exhibited in the first step, $M_w(\tilde{A})$ is either empty or does not meet the subspace Y or meets Y along the singleton $\{0\}$. Hence $M_w(\tilde{A}) \cap \operatorname{aff}(\tilde{A}) = \emptyset$. ∎

Remark 6. It would be sufficient in Theorem 4 to assume that the hypothesis $M_w(A) \cap \operatorname{aff}(A) \neq \emptyset$ works only for subsets A with three or four elements and families (w_a) such that $M_w(A)$ is reduced to a singleton.

As an immediate consequence of Theorem 1 and Theorem 4, we obtain

Corollary 1. *If one of the hull properties (i.e., the equivalent statements of Theorem 1) holds true, then X is an inner product space.*

We consider finally a condition which look like a characterization of inner product spaces related to the Chebyshev radius [6, 9]. Let us introduce a notation. If A is a finite set in X and Z a subspace of X, we let

$$m_X(A) = \inf_{x \in X} \sum_{a \in A} \|x - a\|$$

and

$$m_Z(A) = \inf_{x \in Z} \sum_{a \in A} \|x - a\|.$$

The following result may be compared with (15.1) in [1].

COROLLARY 2. *If, for every two-dimensional subspace Z of X and for every finite set A in Z, we have $m_X(A) = m_Z(A)$, then X is an inner product space.*

Proof. We suppose that X has a three-dimensional subspace Y which is not an inner product space. Because of Corollary 1 there exists a finite subset $\tilde{A}$ in Y such that $M_1^Y(\tilde{A})$, the set of solutions to $\min_{y \in Y} \sum_{a \in \tilde{A}} \|y - a\|$, does not meet $\text{aff}(\tilde{A})$. Therefore $\text{aff}(\tilde{A})$ is an affine subspace of dimension at most two. We may assume, modulo a translation, that $\text{aff}(\tilde{A})$ is included in a two-dimensional linear subspace Z of X. Then $m_Y(\tilde{A}) < m_Z(\tilde{A})$. Since $m_X(\tilde{A}) \leqslant m_Y(\tilde{A})$, we get $m_X(\tilde{A}) < m_Z(\tilde{A})$. ∎

It would be worthwhile to develop the comparison between properties of the Chebyshev radius and Chebyshev centers and properties related to the Fermat problem.

REFERENCES

1. D. AMIR, "Characterizations of Inner Product Spaces," Birkhäuser-Verlag, Basel, 1986.
2. J. BOURGAIN, Strongly exposed points in weakly compact convex sets in Banach spaces, *Proc. Amer. Math. Soc.* **58** (1976), 197–200.
3. G. D. CHAKERIAN AND M. A. GHANDEHARI, The Fermat problem in Minkowski spaces, *Geom. Dedicata* **17** (1985), 227–238.
4. D. CIESLIK, The Fermat-Steiner problem in Minkowski spaces, *Optimization* **19** (1988), 485–489.
5. R. DURIER AND C. MICHELOT, Geometrical properties of the Fermat–Weber problem, *European J. Oper. Res.* **20** (1985), 332–343.
6. A. GARKAVI, On the Chebyshev center and convex hull of a set, *Uspekhi Mat. Nauk USSR* **19** (1964), 139–145.
7. A. GARKAVI AND V. SMATKOV, On the Lamé point and its generalizations in a normed space, *Mat. Sb.* **95** (1974), 267–286.
8. R. C. JAMES, Inner products in normed linear spaces, *Bull. Amer. Math. Soc.* **53** (1947), 559–566.

9. V. KLEE, Circumspheres and inner products, *Math. Scand.* **8** (1960), 363–370.
10. V. KLEE, Sharper approximation of extreme points by far points, *Arch. Math.* **60** (1993), 383–388.
11. H. W. KUHN, On a pair of dual nonlinear problems, *in* "Nonlinear Programming" (J. Abadie, Ed.), pp. 37–54, Wiley, New York, 1967.
12. I. SINGER, Caractérisation des éléments de meilleure approximation dans un espace de Banach quelconque, *Acta Sci. Math.* **17** (1956), 181–189.
13. F. A. VALENTINE, "Convex Sets," McGraw-Hill, New York, 1964.
14. R. E. WENDELL AND P. HURTER, Location theory, dominance and convexity, *Oper. Res.* **21** (1973), 314–321.